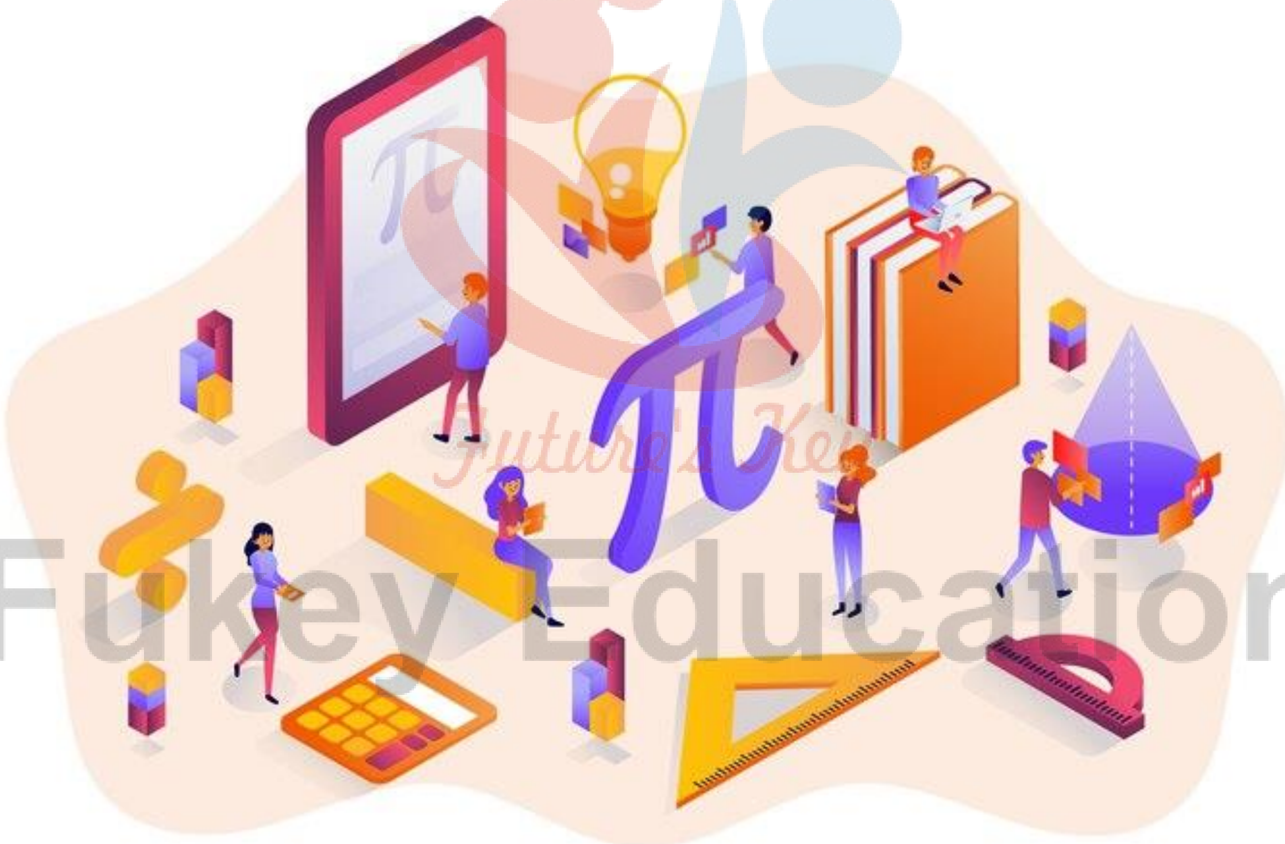
MATHEMATICS

Chapter 9: Differential Equations



DIFFERENTIAL EQUATIONS

1. **Differential Equation:** An equation involving derivatives of the dependent variable with respect to independent variable (variables) is known as a differential equation.

2. **Linear and non-linear differential equation:** A differential equation is said to be linear if unknown function (dependent variable) as its derivative which occurs in the equation, occur only in the first degree, and are not multiplied together. Otherwise, the differential equation is said to be non-linear.

3. **Order:** Order of a differential equation is the order of the highest order derivative occurring in the differential equation.

4. **Degree:** Degree of a differential equation is defined if it is a polynomial equation in its derivatives.

5. Degree (when defined) of a differential equation is the highest power (positive integer only) of the highest order derivative in it.

6. **Solution:** A function which satisfies the given differential equation is called its solution.

7. **General Solution:** The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution.

8. **Particular Solution:** The solution free from arbitrary constants is called particular solution.

9. To form a differential equation from a given function we differentiate the function successively as many times as the number of arbitrary constants in the given function and then eliminate the arbitrary constants.

10. **Variable Separable method:** Variable separable method is used to solve such an equation in which variables can be separated completely i.e., terms containing y should remain with dy and terms containing x should remain with dx.

11. A differential equation which can be expressed in the form $\frac{dy}{dx}\ f(x,y)\ or\ \frac{dx}{dy} g(x,y)$ where, f(x, y) and g (x, y) are homogenous functions of degree zero is called a homogeneous differential equation.

12. A differential equation of the form $\frac{dy}{dx} + Py$ = Q, where P and Q are constants or functions of x only is called a first order linear differential equation.

Class : 12th Maths
Chapter- 9 : Differential Equations

Differential Equations

Definition

An equation involving derivatives of the dependent variable with respect to independent variable (variables) is called a differential equation. If there is only one independent variation, then we call it as an ordinary differential equation. For eg: $2\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0.$

Variable Separation Method

It is used to solve such an equation in which variables can be separated completely. For eg: $y\,dx=x\,dy$ can be solved as $\frac{dx}{x}=\frac{dy}{y}$ Integrating both sides $\log x=\log y+\log c\Rightarrow\frac{x}{y}=c\Rightarrow x=cy$,is the solution.

Order of a Differential Equation

It is the order of the highest order derivative occurring in the Differential Equation For eg: the order of $\frac{dy}{dx}=e^x$ is one and order of $\frac{d^2y}{dx^2}+x=0$ is two.

The order of a Differential Equations representing a family of curves is same as the number of arbitrary constants present in the equation corresponding to the family of curves. For eg: Let the family of curves be $y=mx$, $m=$ constant, then, $y'=m$

$y=y'x\Rightarrow y=\frac{dy}{dx}x\Rightarrow x\frac{dy}{dx}-y=0.$

Degree of a Differential Equation

It is defined if the Differential Equations is a polynomial equation in its derivatives, and is defined as the highest power (positive integer only) of the highest order derivative.

For eg: the degree of $\left(\frac{d^2y}{dx^2}\right)^3+\frac{dy}{dx}=0$ is three

Order and degree (if defined) of a D.E. are always positive integers.

Homogeneous Differential Equation

A Differential Equation which can be expressed in the form $\frac{dy}{dx}=f(x,y)$ or $\frac{dx}{dy}=g(x,y)$, where, $f(x,y)$ and $g(x,y)$ are homogeneous functions of degree zero is called a homogenous Differential Equation

For eg : $(x^2+xy)dy=(x^2+y^2)dx$

To solve this, we substitute $\frac{y}{x}=v\Rightarrow y=vx.$

Linear Differential Equation

A Differential Equation of the form $\frac{dy}{dx}+Py=Q$, where P_1 Q are constants or functions or 'x' only is called a first order linear Differential Equations its solution is $ye^{\int P.dx}=\int Q.e^{\int P.dx}dx+c$. For eg : $\frac{dy}{dx}+3y=2x$ has solution $ye^{\int 3.dx}=\int 2x.e^{\int 3.dx}dx+c\Rightarrow ye^{3x}=2\int xe^{3x}+c.$

Solution of a Differential Equation

A function which satisfies the given Differential Equation is called its solution. The solution which contains as many arbitrary constants as the order of the D.E. is called a general solution and the solution free from arbitrary constants is called particular solution.

For eg: $y=e^x+1$ is a solution of $y''-y'=0$.

Since $y'=e^x$ and $y''=e^x\Rightarrow y''-y'=e^x-e^x=0.$

Formation of Differential Equation

To form a Differential Equation from a given function, we differentiate the function successively as many times as the no. of arbitrary constants in the given function, and then eliminate the arbitrary constants.

For eg: Let the function be $y=ax+b$, then we have to differentiate it two times, since there are 2 arbitrary constants a and b. $\therefore y'=a\Rightarrow y''=0.$ Jhus $y''=0$ is the required Differential Equation.

Important Questions

Multiple Choice questions-

1. The degree of the differential equation:

$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$ is

(a) 3

(b) 2

(c) 1

(d) not defined.

2. The order of the differential equation:

$2x^2 \frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + y = 0$ is

(a) 2

(b) 1

(c) 0

(d) not defined.

3. The number of arbitrary constants in the general solution of a differential equation of fourth order is:

(a) 0

(b) 2

(c) 3

(d) 4.

4. The number of arbitrary constants in the particular solution of a differential equation of third order is:

(a) 3

(b) 2

(c) 1

(d) 0.

5. Which of the following differential equations has $y = c_1 e^x + c_2 e^{-x}$ as the general solution?

(a) $\frac{d^2y}{dx^2} + y = 0$

(b) $\frac{d^2y}{dx^2} - y = 0$

(c) $\frac{d^2y}{dx^2} + 1 = 0$

(d) $\frac{d^2y}{dx^2} - 1 = 0$

6. Which of the following differential equations has y = x as one of its particular solutions?

(a) $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = x$

(b) $\frac{d^2y}{dx^2} + x \frac{dy}{dx} + xy = x$

(c) $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 0$

(d) $\frac{d^2y}{dx^2} + x \frac{dy}{dx} + xy = 0$

7. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

(a) $e^x + e^{-y} = c$

(b) $e^x + e^y = c$

(c) $e^{-x} + e^y = c$

(d) $e^{-x} + e^{-y} = c$.

8. Which of the following differential equations cannot be solved, using variable separable method?

(a) $\frac{dy}{dx} + e^{x+y} + e^{-x+y}$

(b) $(y^2 - 2xy)\, dx = (x^2 - 2xy)\, dy$

(c) $xy \frac{dy}{dx} = 1 + x + y + xy$

(d) $\frac{dy}{dx} + y = 2$.

9. A homogeneous differential equation of the form $\frac{dy}{dx} = h(\frac{x}{y})$ can be solved by making the substitution.

(a) y = vx

(b) v = yx

(c) x = vy

(d) x = v

10. Which of the following is a homogeneous differential equation?

(a) (4x + 6y + 5)dy – (3y + 2x + 4)dx = 0

(b) xy dx – $(x^3 + y^2)$dy = Q

(c) $(x^3 + 2y^2)$ dx + 2xy dy = 0

(d) y^2 dx + $(x^2 - xy - y^2)$dy = 0.

Very Short Questions:

1. Find the order and the degree of the differential equation: $x^2 \frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^4$ (Delhi 2019)

2. Determine the order and the degree of the differential equation: $\left(\frac{dy}{dx}\right)^3 + 2y\frac{d^2y}{dx^2} = 0$ (C.B.S.E. 2019 C)

3. Form the differential equation representing the family of curves: y = b (x + a), where « and b are arbitrary constants. (C.B.S.E. 2019 C)

4. Write the general solution of differential equation:

   $\frac{dy}{dx} = e^{x+y}$ (C.B.S.E. Sample Paper 2019-20)

5. Find the integrating factor of the differential equation:

   $y\frac{dy}{dx} - 2x = y^3e^{-y}$

6. Form the differential equation representing the family of curves y = a sin (3x – b), where a and b are arbitrary constants. (C.B.S.E. 2019C)

Short Questions:

1. Determine the order and the degree of the differential equation:

2. Form the differential equation representing the family of curves: $y = e^{2x}(a + bx)$, where 'a' and 'h' are arbitrary constants. (Delhi 2019)

3. Solve the following differentia equation:

   $\frac{dy}{dx} + y = \cos x - \sin x$ (Outside Delhi 2019)

4. Solve the following differential equation:

   $\frac{dx}{dy} + x = (\tan y + \sec 2y)$. (Outside Delhi 2019 C)

Long Questions:

1. Find the area enclosed by the circle:

   $x^2 + y^2 = a^2$. (N.C.E.R.T.)

2. Using integration, find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. (C.B.S.E. 2018)

3. Find the area bounded by the curves $y = \sqrt{x}$, $2y + 3 = Y$ and Y-axis. (C.B.S.E. Sample Paper 2018-19)

4. Find the area of region:

   $\{(x,y): x^2 + y^2 < 8, x^2 < 2y\}$. (C.B.S.E. Sample Paper 2018-19)

Case Study Questions:

1. If the equation is of the form $\frac{dy}{dx} = py = Q,$ where P, Q are functions of x, then the solution of the differential equation is given by ye $ye^{\int pdx} = \int Q\, e^{\int pdx} dx + c,$ where $e^{\int pdx}$ is called the integrating factor (I.F.).

Based on the above information, answer the following questions.

i. The integrating factor of the differential equation

$\sin x \frac{dy}{dx} + 2y \cos x = 1$ is $(\sin x)^{\lambda}$, where $\lambda =$

a. 0
b. 1
c. 2
d. 3

ii. Integrating factor of the differential equation $(1 - x^2)\frac{dy}{dx} - xy = 1$ is:

a. $-x$

b. $\frac{x}{1+x^2}$

c. $\sqrt{1 - x^2}$

d. $\frac{1}{2}\log(1 - x^2)$

iii. The solution of $\frac{dy}{dx} + y = e^{-x}$, $y(0) = 0$, is:

a. $y = e^x(x - 1)$

b. $y = xe^{-x}$

c. $y = xe^{-x} + 1$

d. $y = (x + 1)e^{-x}$

iv. General solution of $\frac{dy}{dx} + y \tan x = \sec x$ is:

a. $y \sec y = \tan x + c$

b. $y \tan x = \sec x + c$

c. $\tan x = y \tan x + c$

d. $x \sec x = \tan y + c$

v. The integrating factor of differential equation $\frac{dy}{dx} - 3y = \sin 2x$ is:

a. e^{3x}

b. e^{-2x}

c. e^{-3x}

d. xe^{-3x}

2. If the equation is of the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ or $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$, where f(x, y), g(x, y) are homogeneous functions of the same degree in x and y, then put y = vx

And $\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$, so that the dependent variable y is changed to another variable v and then apply variable separable method.

Based on the above information, answer the following questions.

i. The general solution of $x^2 \frac{dy}{dx} = x^2 + xy + y^2$ is:

a. $\tan^{-1} \frac{x}{y} = \log|x| + c$

b. $\tan^{-1} \frac{y}{x} = \log|x| + c$

c. $y = x \log|x| + c$

d. $x = y \log|y| + c$

ii. Solution of the differential equation $2xy \frac{dy}{dx} = x^2 + 3y^2$ is:

a. $x^3 + y^2 = cx^2$

b. $\frac{x^2}{2} + \frac{y^3}{3} = y^2 + c$

c. $x^2 + y^3 = cx^2$

d. $x^2 + y^2 = cx^3$

iii. General solution of the differential equation $(x^2 + 3xy + y^2)\,dx - x^2\,dy = 0$ is:

a. $\frac{x+y}{y} - \log x = c$

b. $\frac{x+y}{y} + \log x = c$

c. $\frac{x}{x+y} - \log x = c$

d. $\frac{x}{x+y} + \log x = c$

iv. General solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}\left\{\log\left(\frac{y}{x}\right) + 1\right\}$ is:

a. $\log(xy) = c$

b. $\log y = cx$

c. $\log \frac{y}{x} = cx$

d. $\log x = cy$

v. Solution of the differential equation $\left(x\frac{dy}{dx} - y\right)e^{\frac{y}{x}} = x^2 \cos x$ is:

a. $e^{\frac{y}{x}} - \sin x = c$

b. $e^{\frac{y}{x}} + \sin x = c$

c. $e^{\frac{-y}{x}} - \sin x = c$

d. $e^{\frac{-y}{x}} + \sin x = c$

Answer Key-

Multiple Choice questions-

1. Answer: (a) 3
2. Answer: (a) 2
3. Answer: (d) 4.

4. Answer: (d) 0.

5. Answer: (b) $\frac{d^2y}{dx^2} - y = 0$

6. Answer: (c) $\frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 0$

7. Answer: (a) $e^x + e^{-y} = c$

8. Answer: (b) $(y^2 - 2xy)\,dx = (x^2 - 2xy)\,dy$

9. Answer: (c) $x = vy$

10. Answer: (d) $y^2\,dx + (x^2 - xy - y^2)dy = 0$.

Very Short Answer:

1. Solution: Here, order = 2 and degree = 1.

2. Solution: Order = 2 and Degree = 1.

3. Solution:

   We have: y= b(x + a) ...(1)

   Diff. w.r.t. x, b.

   Again diff. w.r.t. x, $\frac{d^2y}{dx^2} = 0$,

   which is the reqd. differential equation.

4. Solution:

   We have: $\frac{dy}{dx} = e^{x+y}$

   $\Rightarrow e^{-y}\,dy = e^x\,dx$ [Variables Separable

   Integrating, $\int e^{-y}dy + c = \int e^x dx$

   $\Rightarrow -e^{-y} + c = e^x$

   $\Rightarrow e^x + e^{-y} = c$.

5. Solution:

   The given equation can be written as.

$$\frac{dy}{dx} - \frac{2x}{y} = y^2e^{-y}.$$

$$\therefore \quad \text{I.F.} = e^{-\int \frac{2}{y}dy}$$

$$= e^{-2\log|y|} = e^{\log\frac{1}{y^2}} = \frac{1}{y^2}$$

6. Solution:

We have: y – a sin (3x – b) ...(1)

Diff. W.r.t y $\frac{dy}{dx}$ = a cos (3x – b) .3

= 3a cos (3x – b)

$\frac{d^2y}{dx^2}$ = -3a sin (3x – b) 3

= -9a sin (3x – b)

= -9y [Using (1)]

$\frac{d^2y}{dx^2}$ + 9y = 0,m

which in the reqd. differential equation.

Short Answer:

1. Solution: Order = 2 and Degree = 1.
2. Solution:

We have: y = e^{2x} (a + bx) ...(1)

Diff. w.r.t. x, $\frac{dy}{dx}$ = e^{2x} (b) + 2e2x (a + bx)

$\Rightarrow \frac{dy}{dx}$ = be^{2x} + 2y .... (2)

Again diff. w.r.t. x,

$\frac{d^2y}{dx^2}$ = $2be^{2x} + 2^{2x}$

$\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx} - 2y\right) + \frac{dy}{dx}$

[Using (2)]

Hence, $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$, which is the reqd. differential equation.

3. Solution:

The given differential equation is:

$\frac{dy}{dx} + y = \cos x - \sin x$ dx Linear Equation

$\therefore$ I.F. = $e^{\int 1dx}$ = ex

The solution is :

$y.e^x = \int (\cos x - \sin x)\, e^x\, dx + C$

$\Rightarrow y.e^x = e^x \cos x + C$

or $y = \cos x + C e^{-x}$

4. Solution:

The given differential equation is:

$\frac{dx}{dy} + x = (\tan y + \sec^2 y)$.

Linear Equation

$\because$ I.F. = ∫ldy = ey

$\therefore$ The solution is:

x. ey = $\int$ ey (tan y + $\sec^2$ y)dy + c

$\Rightarrow$ x. ey = ey tan y + c

= x = tan y + c e^{-y}, which is the reqd. solution.

Long Answer:

1. Solution:

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{\frac{y^2}{x^2} - 1}{\frac{2y}{x}}$$

Put $\frac{y}{x} = v$

$\Rightarrow$ $y = vx$ and so $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ $v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$

$\Rightarrow$ $\frac{x\,dv}{dx} = -\frac{(1+v^2)}{2v}$

$\Rightarrow$ $\int \frac{dx}{x} = -\int \frac{2v\,dv}{1+v^2}$

$\log x = -\log(1 + v^2) + \log C$

$x(1 + v^2) = C$

$x\left(1 + \frac{y^2}{x^2}\right) = C$

$x^2 + y^2 = C.$

2. Solution:

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2}$$

$$\text{I.F.} = e^{\int \frac{2x}{1+x^2}} + e^{\log(1+x^2)} = (1+x^2).$$

Solution is $y(1 + x^2) = \int \frac{1}{1+x^2}dx$

$= \tan^{-1} x + C$

When $y = 0$, $x = 1$,

then $0 = \frac{\pi}{4} + C$

$C = \frac{\pi}{4}$

$\therefore y(1 + x^2) = \tan^{-1} x - \frac{\pi}{4}$

i.e, $y = \frac{\tan^{-1} x}{1+x^2} - \frac{\pi}{4(1+x^2)}$

3. Solution:

We have: $y = ae^{bx+5} + 5$ ...(1)

Diff. w.r.t. x, $\frac{dy}{dx} = ae^{bx+5}.(b)$

$\frac{dy}{dx} = dy$ ......(2) [Using (1)]]

Again diff. w.r.t x.,

$\frac{d^2y}{dx^2} = b\frac{dy}{dx}$ ....(3)

Dividing (3) by (2),

$$\frac{\frac{d^2y}{dx^2}}{\frac{dy}{dx}} = \frac{\frac{dy}{dx}}{y}$$

$$\Rightarrow \quad y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0,$$

which is the required differential equation.

4. Solution:

The given differential equation is:

$x\,dx - ye^y\sqrt{1+x^2}\;dy = 0$

$\Rightarrow \dfrac{x\,dx}{\sqrt{1+x^2}} - ye^y\,dy = 0$ [*Variables Separable*]

Integrating, $\int \dfrac{x\,dx}{\sqrt{1+x^2}} - \int ye^y\,dy = c$ ...(1)

Now, $\int \dfrac{x}{\sqrt{1+x^2}}dx = \dfrac{1}{2}\int (1+x^2)^{-1/2}(2x)\,dx$

$= \dfrac{1}{2}\dfrac{(1+x^2)^{1/2}}{1/2} = \sqrt{1+x^2}$.

And, $\int y\,e^y dy = y \cdot e^y - \int (1)\,e^y dy$

[*Integrating by parts*

$= ye^y - e^y$.

$\therefore$ From (1), $\sqrt{1+x^2} - (ye^y - e^y) = c$

$\Rightarrow \sqrt{1+x^2} = c + e^y(y-1)$ ...(2)

When x = 0, y = 1, ∴ 1 = c + c(0) ⇒ c = 1.

Putting in (2), $\sqrt{1+x^2} = 1 + e^y(y-1)$,

which is the reqd. particular solution.

Case Study Answers:

1. Answer :

i. (c) 2

Solution:

The given differential equation can be written as $\frac{dy}{dx} + 2y\cot x = \operatorname{cosec} x$

$\therefore\ I.F = e^{\int 2\cot x dx} = e^{2\log|\sin x|} = (\sin x)^2$

$\therefore\ \lambda = 2$

ii. (c) $\sqrt{1-x^2}$

Solution:

We have, $(1-x^2)\frac{dy}{dx}-xy=1$

$\Rightarrow \frac{dy}{dx}-\frac{x}{1-x^2}\cdot y=\frac{1}{1-x^2}$

$\therefore$ I.F. $=e^{-\int\frac{x}{1-x^2}dx}=e^{\frac{1}{2}\int\frac{-2x}{1-x^2}dx}$

$=e^{\frac{1}{2}\log(1-x^2)}=e^{\log(1-x^2)^{\frac{1}{2}}}=\sqrt{1-x^2}$

iii. (b) $y=xe^{-x}$

Solution:

We have, $\frac{dy}{dx}+y=e^{-x}$

It is a linear differential equation with I.F. $=e^{\int dx}=e^x$

Now, solution is $y\cdot e^x=\int e^{-x}dx+c$

$\Rightarrow ye^x=\int dx+c$

$\Rightarrow ye^x=x+c$

$\Rightarrow y=xe^{-x}+ce^{-x}$

$\because y(0)=0\Rightarrow c=0$

$\therefore y=xe^{-x}$

iv. (a) $y \sec y = \tan x + c$

Solution:

We have, $\frac{dy}{dx} + y \tan x = \sec x$

It is a linear differential equation with,

$$I.F. = e^{\int \tan x dx} = e^{\log|\sec x|} = \sec x$$

Now, solution is $y \sec x = \int \sec^2 x\, dx + c$

$$\Rightarrow y \sec x = \tan x + c$$

v. (c) e^{-3x}

Solution:

We have, $\frac{dy}{dx} 3y = \sin 2x$

It is a linear differential equation with,

$$I.F. = e^{\int -3dx} = e^{-3x}$$

2. Answer :

i. (b) $\tan^{-1} \frac{y}{x} = \log|x| + c$

Solution:

We have, $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$

Put y = vx and $\frac{dy}{dx} = v+x\frac{dv}{dx}$

$$\therefore\ v+x\frac{dv}{dx} = \frac{x^2+x\times vx+v^2x^2}{x^2} = 1 + v + v^2$$

$$\Rightarrow x\frac{dv}{dx} = 1 + v^2 \Rightarrow \int \frac{dv}{1+v^2} = \int \frac{dx}{x} + c$$

$$\Rightarrow \tan^{-1} v = \log|x| + c$$

$$\Rightarrow \tan^{-1} \frac{y}{x} = \log|x| + c$$

ii. (d) $x^2 + y^2 = cx^3$

Solution:

We have,

$$2xy\frac{dy}{dx} = x^2 + 3y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$$

Put y = vx and $\frac{dy}{dx} = v+x\frac{dv}{dx}$

$$\therefore\ v+x\frac{dv}{dx} = \frac{x^2+3v^2x^2}{2vx^2}$$

$$\Rightarrow\ x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v$$

$$\Rightarrow\ x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

$$\Rightarrow \int \frac{2v}{1+v^2}dv = \int \frac{dx}{x} + \log c$$

$$\Rightarrow \log|1+v^2| = \log|x| + \log|c|$$

$$\Rightarrow \log|v^2+1| = \log|xc|$$

$$\Rightarrow v^2 + 1 = xc \Rightarrow \frac{y^2}{x^2} + 1 = xc$$

$$\Rightarrow x^2 + y^2 = x^3c$$

iii. (d) $\frac{x}{x+y}+\log x=c$

Solution:

We have,

$(x^2 + 3xy + y^2)\, dx - x^2\, dy = 0$

$\Rightarrow \frac{x^2+3xy+y^2}{x^2}=\frac{dy}{dx}$

Put y= vx and $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore \frac{x^2+3x^2v+x^2v^2}{x^2}=\left(v+x\frac{dv}{dx}\right)$

$\Rightarrow 1+3v+v^2=v+x\frac{dv}{dx}$

$\Rightarrow 1+2v+v^2=x\frac{dv}{dx}$

$\Rightarrow \int\frac{dx}{x}-\int(v+1)^{-2}=dv=c$

$\log x+\frac{1}{v+1}=c$

$\Rightarrow \log x+\frac{x}{x+y}=c$

iv. (c) $\log \frac{y}{x} = cx$

Solution:

We have, $\frac{dy}{dx} = \frac{y}{x}\left\{\log\left(\frac{y}{x}\right) + 1\right\}$

Put y = vx and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore\ v + x\frac{dv}{dx} = v\{\log(v + 1\}$

$\Rightarrow x\frac{dv}{dx} = v\log v$

$\Rightarrow \int \frac{dv}{v\log v} = \int \frac{dx}{x} \Rightarrow \log|\log v| = \log|x| + \log|c|$

$\Rightarrow \log\left(\frac{y}{x}\right) = cx$

v. (a) $e^{\frac{y}{x}} - \sin x = c$

Solution:

We have, $\left(x\frac{dy}{dx} - y\right)e^{\frac{y}{x}} = x^2\ \cos x$

$\Rightarrow \left(\frac{dy}{dx} - \frac{y}{x}\right)e^{\frac{y}{x}} = x\ \cos x$

Put y = vx and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \left(v + x\frac{dv}{dx} - v\right)e^{v} = x\cos x$

$\Rightarrow xe^{v}\frac{dv}{dx} = x\cos x$

$\Rightarrow \int e^{v}dv = \int \cos x\, dx$

$\Rightarrow e^{v} = \sin x + c$

$\Rightarrow\ e^{\frac{y}{x}} - \sin x = c$