MATHEMATICS

Chapter 8: Application of Integrals

APPLICATION OF INTEGRALS

1. **Elementary area:** The area is called elementary area which is located at any arbitary position within the region which is specified by some value of x between a and b.

2. The area of the region bounded by the curve y = f (x), x-axis and the lines x = a and x = b (b > a) is given by the formula: $Area = \int_z^b vdx = \int_z^b f(x)dx$

3. The area of the region bounded by the curve $x = \theta(y)$, y-axis and the lines y = c, y = d is given by the formula: $Area = \int_c^b xdy = \int_c^b \theta(y)dy$

4. The area of the region enclosed between two curves y = f (x), y = g (x) and the lines x = a, x = b is given by the formula, Area = $\int_a^b [f(x) - g(x)]dx$, where $f(x) \geq g(x)$ in [a, b].

5. If $f(x) \geq g(x)$ in [a, c] and $f(x) \leq g(x)$ in [c, b], a < c < b, then we write the areas as:

$$Area = \int_a^b [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$$

Class : 12th Maths
Chapter- 8 : Applications of Integrals

The area of the region enclosed between two curves $y=f(x), y=g(x)$ and the lines $x=a$, $x=b$ is given by

$$A=\int_a^b[f(x)-g(x)]dx, \text{ where } f(x)\geq g(x) \text{ in } [a,b]$$

For eg: To find the area of the region bounded by the two parabolas $y=x^2$ and $y^2=x$

(0,0) and (1,1) are points of intersection of $y=x^2$ and y^2-x and, $y^2=x \Rightarrow y=\sqrt{x}=f(x)$, and $y=x^2=g(x)$, where $f(x)\geq g(x)$ in $[0,1]$.

$$\text{Area, } A=\int_0^1[f(x)-g(x)]dx$$

$$=\int_0^1\left[\sqrt{x}-x^2\right]dx$$

$$=\left[\frac{2}{3}x^{3/2}-\frac{x^3}{3}\right]_0^1$$

$$=\frac{2}{3}-\frac{1}{3}=\frac{1}{3} \text{ Sq. units.}$$

if $f(x)\geq g(x)$ in $[a,c]$ and $f(x)\leq g(x)$ in $[c, \ b]$, $a<c<b$, then the area is

$$A=\int_a^c[f(x)-g(x)]dx+\int_c^b[g(x)-f(x)]dx$$

Area between two curves

Applications of Integrals

Area under simple curves

The area of the region bounded by the curve $x=f(y)$, $y-axis$ and the lines $y=c$ and $y=d\,(d>c)$ is given by $A=\int_c^d x\,dy$ or $\int_c^d f(y)dy$.

For eg : the area bounded by $x=y^3$, $y-axis$ in the I quadrant and the lines $y=1$ and $y=2$ is

$$A=\int_1^2 x\,dy=\int_1^2 y^3\,dy=\left[\frac{1}{4}y^4\right]_1^2=\frac{1}{4}(2^4-1^4)=\frac{15}{4} \text{ Sq. units}$$

The area of the region bounded by the curve $y=f(x)$, x- $axis$ and the lines $x=a$ and $x=b\,(b>a)$ is given by

$$A=\int_a^b y\,dx \text{ or } \int_a^b f(x)dx.$$

For eg : the area bounded by $y=x^2$, x- $axis$ in I quadrant and the lines $x=2$ and $x=3$ is -

$$A=\int_2^3 y\,dx=\int_2^3 x^2\,dx=\left[\frac{x^3}{3}\right]_2^3=\frac{1}{3}(27-8)=\frac{19}{3} \text{ Sq.units.}$$

Important Questions

Multiple Choice questions-

1. Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is

(a) π

(b) $\frac{\pi}{2}$

(c) $\frac{\pi}{3}$

(d) $\frac{\pi}{4}$

2. Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$ is

(a) 2

(b) $\frac{9}{4}$

(c) $\frac{9}{3}$

(d) $\frac{9}{2}$

3. Smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$ is

(a) 2 (π – 2)

(b) π – 2

(c) 2π – 1

(d) 2 (π + 2).

4. Area lying between the curves $y^2 = 4x$ and $y = 2$ is:

(a) $\frac{2}{3}$

(b) $\frac{1}{3}$

(c) $\frac{1}{4}$

(d) $\frac{3}{4}$

5. Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is

(a) -9

(b) $-\frac{15}{4}$

(c) $\frac{15}{4}$

(d) $\frac{17}{4}$

6. The area bounded by the curve $y = x|x|$, x-axis and the ordinates $x = -1$ and $x = 1$ is given by

(a) 0

(b) $-\frac{1}{3}$

(c) $\frac{2}{3}$

(d) $\frac{4}{3}$

7. The area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$ is

(a) $\frac{4}{3}(4\pi - \sqrt{3})$

(b) $\frac{1}{3}(4\pi + \sqrt{3})$

(c) $\frac{2}{3}(8\pi - \sqrt{3})$

(d) $\frac{4}{3}(8\pi + \sqrt{3})$

8. The area enclosed by the circle $x^2 + y^2 = 2$ is equal to

(a) 4π sq. units

(b) $2\sqrt{2}\,\pi$ sq. units

(c) $4\pi^2$ sq. units

(d) 2π sq. units.

9. The area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to

(a) $\pi^2 ab$

(b) πab

(c) $\pi a^2 b$

(d) πab^2.

10. The area of the region bounded by the curve $y = x^2$ and the line $y = 16$ is

(a) $\frac{32}{3}$

(b) $\frac{256}{3}$

(c) $\frac{64}{3}$

(d) $\frac{128}{3}$

Very Short Questions:

1. Find the area of region bounded by the curve $y = x^2$ and the line $y = 4$.
2. Find the area bounded by the curve $y = x^3$, $x = 0$ and the ordinates $x = -2$ and $x = 1$.
3. Find the area bounded between parabolas $y^2 = 4x$ and $x^2 = 4y$.
4. Find the area enclosed between the curve $y = \cos x$, $0 \leq x \leq \frac{\pi}{4}$ and the co-ordinate axes.
5. Find the area between the x-axis curve $y = \cos x$ when $0 \leq x < 2$.
6. Find the ratio of the areas between the center $y = \cos x$ and $y = \cos 2x$ and x-axis for $x = 0$ to

   $x = \frac{\pi}{3}$
7. Find the areas of the region:

   $\{(x,y): x^2 + y^2 \leq 1 \leq x + 4\}$

Long Questions:

1. Find the area enclosed by the circle:

   $x^2 + y^2 = a^2$. (N.C.E.R.T.)
2. Using integration, find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. (C.B.S.E. 2018)

3. Find the area bounded by the curves $y = \sqrt{x}$, $2y + 3 = Y$ and Y-axis. (C.B.S.E. Sample Paper 2018-19)

4. Find the area of region:

   $\{(x,y): x^2 + y^2 < 8, x^2 < 2y\}$. (C.B.S.E. Sample Paper 2018-19)

Case Study Questions:

1. Ajay cut two circular pieces of cardboard and placed one upon other as shown in figure. One of the circle represents the equation $(x - 1)^2 + y^2 = 1$, while other circle represents the equation $x^2 + y^2 = 1$.

Based on the above information, answer the following questions.

i. Both the circular pieces of cardboard meet each other at

a. $x = 1$

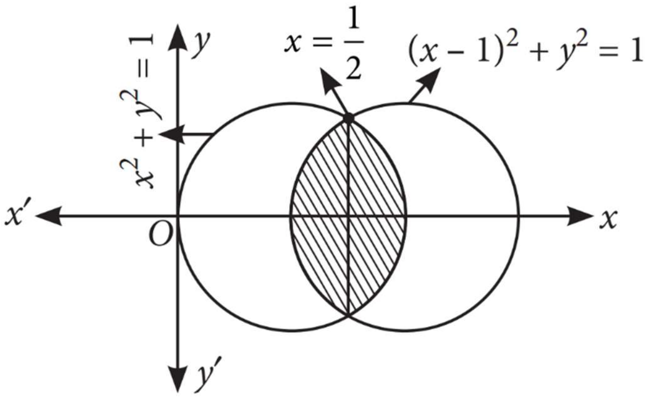
b. $x = \frac{1}{2}$

c. $x = \frac{1}{3}$

d. $x = \frac{1}{4}$

ii. Graph of given two curves can be drawn as.

a.



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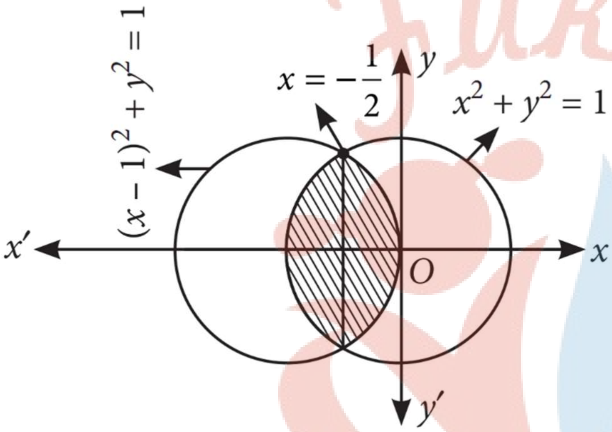
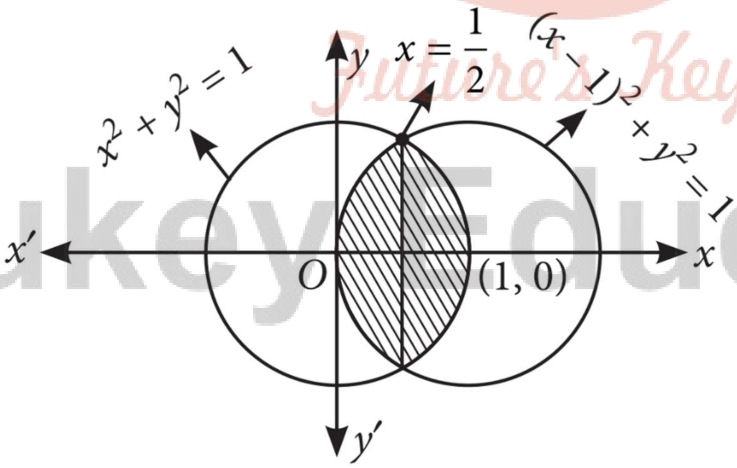


c.



d. None of these

iii. Value of $\int_{0}^{\frac{1}{2}} \sqrt{1-(x-1)^2}dx$ is.

a. $\frac{\pi}{6}-\frac{\sqrt{3}}{8}$

b. $\frac{\pi}{6}+\frac{\sqrt{3}}{8}$

c. $\frac{\pi}{2}+\frac{\sqrt{3}}{4}$

d. $\frac{\pi}{2}-\frac{\sqrt{3}}{4}$

iv. Value of $\int_{\frac{1}{2}}^{1} \sqrt{1-x^2}dx$ is.

a. $\frac{\pi}{6}+\frac{\sqrt{3}}{4}$

b. $\frac{\pi}{6}+\frac{\sqrt{3}}{8}$

c. $\frac{\pi}{6}-\frac{\sqrt{3}}{8}$

d. $\frac{\pi}{2}-\frac{\sqrt{3}}{4}$

V. Area of hidden portion of lower circle is.

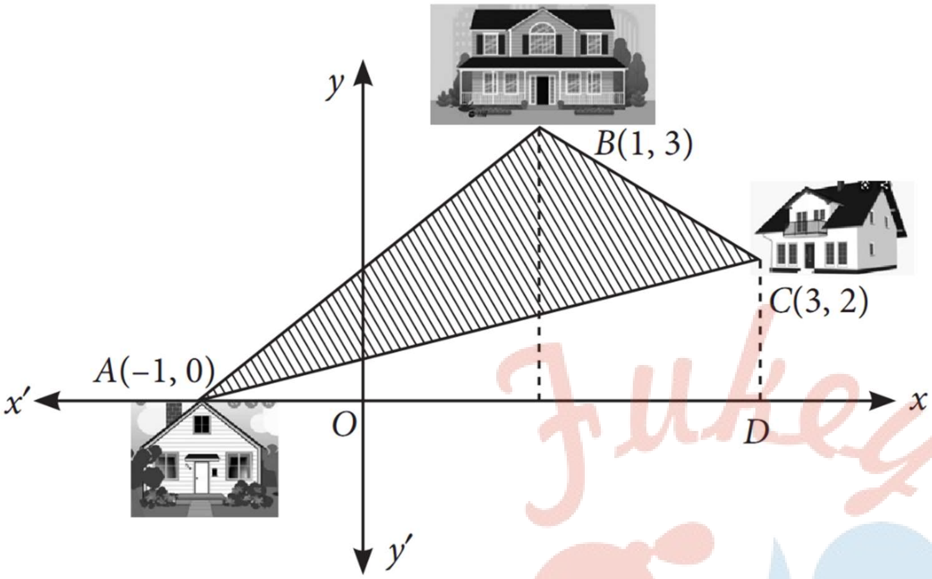
a. $\left(\frac{2\pi}{3}+\frac{\sqrt{3}}{2}\right)$ sq.units

b. $\left(\frac{\pi}{3}-\frac{\sqrt{3}}{8}\right)$ sq.units

c. $\left(\frac{\pi}{3}+\frac{\sqrt{3}}{8}\right)$ sq.units

d. $\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$ sq.units

2. Location of three houses of a society is represented by the points A(-1, 0), B(1, 3) and C(3, 2) as shown in figure.



Based on the above information, answer the following questions

(i) Equation of line AB is.

a. $y = \frac{3}{2}(x + 1)$

b. $y = \frac{3}{2}(x - 1)$

c. $y = \frac{1}{2}(x + 1)$

d. $y = \frac{1}{2}(x - 1)$

(ii) Equation of line BC is.

a. $y = \frac{1}{2}x - \frac{7}{2}$

b. $y = \frac{3}{2}x - \frac{7}{2}$

c. $y = \frac{-1}{2}x + \frac{7}{2}$

d. $y = \frac{3}{2}x + \frac{7}{2}$

(iii) Area of region ABCD is.

a. 2 sq. units
b. 4 sq. units
c. 6 sq. units
d. 8 sq. units

(iv) Area of $\triangle$ADC is,

a. 4 sq. units
b. 8 sq. units
c. 16 sq. units
d. 32 sq. units

(v)Area of $\triangle$ABC is.

a. 3 sq. units
b. 4 sq. units
c. 5 sq. units
d. 6 sq. units

Answer Key-

Multiple Choice questions-

1. Answer: (a) π
2. Answer: (a) 2
3. Answer: (b) $\pi - 2$
4. Answer: (b) $\frac{1}{3}$
5. Answer: (b) $-\frac{15}{4}$
6. Answer: (c) $\frac{2}{3}$
7. Answer: (c) $\frac{2}{3}(8\pi - \sqrt{3})$
8. Answer: (d) 2π sq. units.
9. Answer: (b) πab
10. Answer: (b) $\frac{256}{3}$

Very Short Answer:

1. Solution: $\frac{32}{2}$ sq. units.

2. Solution: $\frac{17}{4}$ sq. units.

3. Solution: $\frac{16}{3}$ sq. units.

4. Solution: $\frac{1}{2}$ sq. units.

5. Solution: 4 sq. units

6. Solution: 2 : 1.

7. Solution: $\frac{1}{2}$ (π – 1) sq. units.

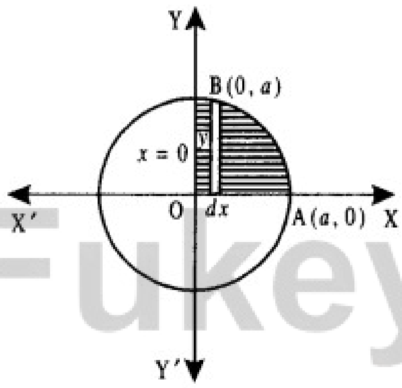
Long Answer:

1. Solution:

The given circle is

$x^2 + y^2 = a^2$ ………… (1)

This is a circle whose center is (0,0) and radius 'a'.



Area of the circle=4 x (area of the region OABO, bounded by the curve, x-axis and ordinates x = 0, x = a)

[∵ Circle is symmetrical about both the axes]

$= 4\int_0^a y dx$ [Taking vertical strips] o

$= 4\int_0^a \sqrt{a^2 - x^2} dx$

[∵ (1) ⇒ y = ± $\sqrt{a^2 - x^2}$

But region OABO lies in 1st quadrant, ∴ y is + ve]

$$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\left\{\frac{a}{2}(0) + \frac{a^2}{2}\sin^{-1}(1)\right\} - \{0+0\}\right]$$

$$= 4\left(\frac{a^2}{2}\cdot\frac{\pi}{2}\right) = \pi a^2 \text{ sq. units.}$$

2. Solution:

We have:

$y = x$ ...(l)

and $x^2 + y^2 = 32$ ...(2)

(1) is a st. line, passing through (0,0) and (2) is a circle with centre (0,0) and radius $4\sqrt{2}$ units. Solving (1) and (2) :

Putting the value of y from (1) in (2), we get:

$x^2 + x^2 = 32$

$2x^2 = 32$

$x^2 = 16$

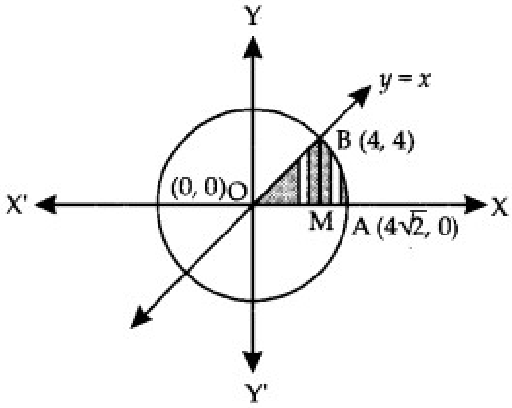
$x = 4$.

[∵ region lies in first quadrant]

Also, $y = 4$

Thus, the line (1) and the circle (2) meet each other at B (4,4), in the first quadrant.

Draw BM perp. to x – axis.



∴ Reqd. area = area of the region OMBO + area of the region BMAB ...(3)

Now, area of the region OMBO

$$= \int_0^4 y\,dx \qquad \text{[Taking vertical strips]}$$

$$= \int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4 = \frac{1}{2}(16-0) = 8.$$

Again, area of the region BMAB

$$= \int_4^{4\sqrt{2}} y\,dx \qquad \text{[Taking vertical strips]}$$

$$= \int_4^{4\sqrt{2}} \sqrt{32-x^2}\,dx$$

[$\because$ $y^2 = 32 - x^2 \Rightarrow y = \sqrt{32-x^2}$, *taking +ve sign, as it lies in Ist quadrant*]

$$= \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2}\,dx$$

$$= \left[\frac{x\sqrt{32-x^2}}{2} + \frac{32}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= \left\{\frac{1}{2}4\sqrt{2}\times 0 + \frac{32}{2}\sin^{-1}(1)\right\} - \left\{\frac{4}{2}\sqrt{32-16} + \frac{32}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right\}$$

$$= 0 + 16\left(\frac{\pi}{2}\right) - \left(2\times 4 + 16\times\frac{\pi}{4}\right)$$

$= 8\pi - (8 + 4\pi) = 4\pi - 8$

∴ From (3),

Required area = $8 + (4\pi - 8) = 4\pi$ sq. units.

3. Solution:

The given curves are

$y = \sqrt{x}$ ....(1)

and $2y + 3 = x$ ...(2)

Solving (1) and (2), we get;

$\sqrt{2y+3} = y$

Squaring, $2y + 3 = y2$

$\Rightarrow y^2 2 - 2y - 3 = 0$

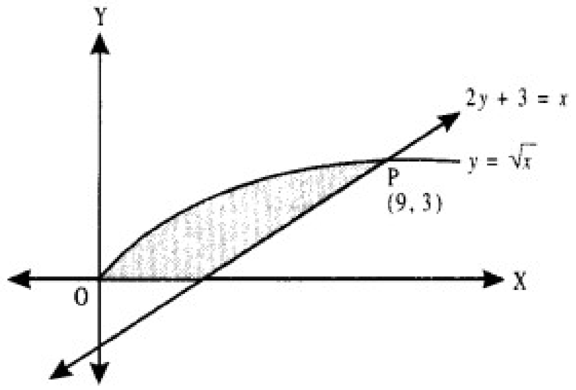
$\Rightarrow (y + 1)(y-3) = 0 \Rightarrow y = -1, 3$

$\Rightarrow y = 3$ [$\because y > 0$]

Putting in (2),

$x = 2(3) + 3 = 9$.

Thus, (1) and (2) intersects at (9, 3).



$$\therefore \quad \text{Reqd. Area} = \int_0^3 (2y+3)dy - \int_0^3 y^2 dy$$

$$= [y^2 + 3y]_0^3 - \left[\frac{y^3}{3}\right]_0^3$$

$$= (9+9) - \left(\frac{27}{3}\right)$$

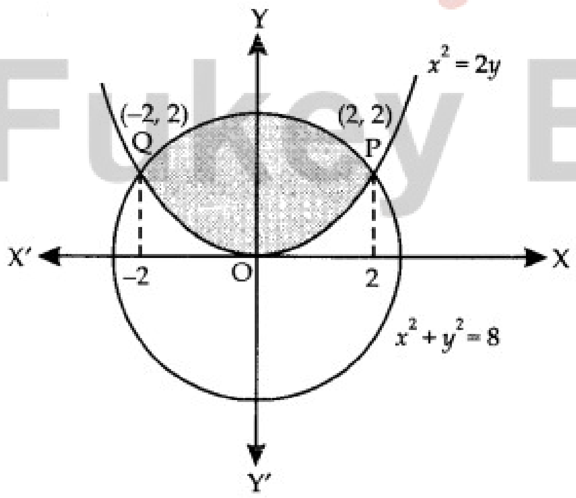
$$= 9 + 9 - 9 = 9 \text{ sq. units.}$$

4. Solution:

The given curves are;

$x^2 + y^2 = 8$ .... (1)

$x^2 = 2y$ .... (2)



Solving (1) and (2):

$8 - y^2 = 2y$

$\Rightarrow y^2 + 2y - 8 = 0$

$\Rightarrow (y + 4)(y - 2) = 0$

$= y = -4, 2$

$\Rightarrow y = 2.\ [\because y > 0]$

Putting in (2), $x^2 = 4$

$\Rightarrow x = -2$ or 2.

Thus, (1) and (2) intersect at P(2, 2) and Q(-2, 2).

$$\therefore \text{Required area} = \int_{-2}^{2} \sqrt{8-x^2}\,dx - \int_{-2}^{2} \frac{x^2}{2}\,dx$$

$$= 2\left[\int_0^2 \sqrt{(2\sqrt{2})^2 - x^2}\,dx - \int_0^2 \frac{x^2}{2}\,dx\right]$$

$$= 2\left[\frac{x\sqrt{8-x^2}}{2} + \frac{8}{2}\sin^{-1}\left(\frac{x}{2\sqrt{2}}\right)\right]_0^2 - \frac{1}{3}[x^3]_0^2$$

$$= 2\left[2 + 4\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 0\right] - \frac{1}{3}[8-0]$$

$$= 4 + 8\left(\frac{\pi}{4}\right) - \frac{8}{3} = \left(2\pi + \frac{4}{3}\right)\text{sq. units}.$$

Case Study Answers:

1. Answer :

i. (b) $x = \frac{1}{2}$

Solution:

We have, $(x - 1)^2 + y^2 = 1$

$$\Rightarrow y = \sqrt{1-(x-1)^2}$$

Also $x^2 + y^2 = 1$

$$\Rightarrow y = \sqrt{1-x^2}$$

From (i) and (ii), we get

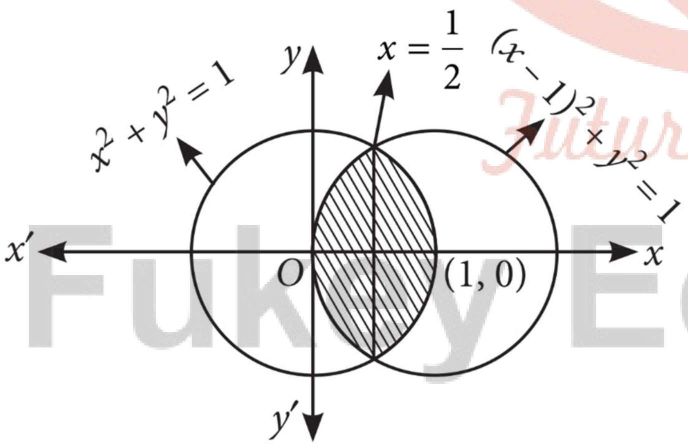
$$\sqrt{1-(x-1)^2} = \sqrt{1-x^2}$$

$$\Rightarrow (x-1)^2 = x^2$$

$$\Rightarrow 2x = 1$$

$$\Rightarrow x = \frac{1}{2}$$

ii. (c)



iii. (a) $\frac{\pi}{6} - \frac{\sqrt{3}}{8}$

Solution:

$$\left[\int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-1}{1}\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{2}-1\right)\sqrt{1-\frac{1}{4}} + \frac{1}{2} + \sin^{-1}\left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)$$

$$-\frac{1}{2}\sin^{-1}$$

$$= \left[\frac{-1}{4}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{\pi}{6} + 0 + \frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{\sqrt{3}}{8} - \frac{\pi}{12} + \frac{\pi}{4}$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

iv. (c) $\frac{\pi}{6} - \frac{\sqrt{3}}{8}$

$$\int_{\frac{1}{2}}^{1} \sqrt{1-x^2}dx = \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{\frac{1}{2}}^{1}$$

$$= 0 + \frac{1}{2}\sin^{-1}(1) - \frac{1}{4}\sqrt{1-\frac{1}{4}} - \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{4} - \frac{\sqrt{3}}{8} - \frac{\pi}{12} = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

v. (d) $\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ sq.units

Solution:

$$= 2\left[\int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}dx + \int_{\frac{1}{2}}^{1} \sqrt{1-x^2}dx\right]$$

$$= 2\left[\frac{\pi}{6} - \frac{\sqrt{3}}{8} + \frac{\pi}{6} - \frac{\sqrt{3}}{8}\right]$$

$$= 2\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ sq.units}$$

2. Answer :

i. (a) $y = \frac{3}{2}(x+1)$

Solution:

Equation of line AB is. $y - 0 = \frac{3-0}{1+1}(x+1) \Rightarrow y = \frac{3}{2}(x+1)$

ii. (c) $y = \frac{-1}{2}x + \frac{7}{2}$

Solution:

Equation of line BC is $y - 3 = \frac{2-3}{3-1}(x+1)$

$$\Rightarrow y = -\frac{1}{2}x + \frac{1}{2} + 3 \Rightarrow y = \frac{-1}{2}x + \frac{7}{2}$$

iii. (d) 8 sq. units

Solution:

Area of region ABCD = Area of $\triangle ABE +$ Area of region BCDE

$$= \int_{-1}^{1} \frac{3}{2}(x+1)dx + \int_{1}^{3}\left(\frac{-1}{2}x + \frac{7}{2}\right)dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{1} + \left[\frac{-x^2}{4} + \frac{7}{2}x\right]_{1}^{3}$$

$$\frac{3}{2}\left[\frac{1}{2} + 1 - \frac{1}{2} + 1\right] + \left[\frac{-9}{4} + \frac{21}{2} + \frac{1}{4} - \frac{7}{2}\right]$$

$$= 3 + 5 = 8 \text{ sq.units}$$

iv. (a) 4 sq. units

Solution:

Equation of line AC is $y - 0 = \frac{2-0}{3+1}(x+1)$

$$\Rightarrow y = \frac{1}{2}(x+1)$$

$$\therefore \text{Area of } \triangle ADC = \int_{-1}^{3} \frac{1}{2}(x+1)dx = \left[\frac{x^2}{4} + \frac{1}{2}x\right]_{-1}^{3}$$

$$= \frac{9}{4} + \frac{3}{2} - \frac{1}{4} + \frac{1}{2} = 4 \text{ sq.units}$$

v. (b) 4 sq. units

Solution:

Area of $\triangle ABC$ $=$ Area of region ABCD - Area of $\triangle ACD = 8 - 4 = 4$ sq.units