MATHEMATICS

Chapter 1: Relation and Function

RELATIONS AND FUNCTIONS

Top Concepts in Relations

1. Introduction to Relation and no. of relations

- A relation R between two non-empty sets A and B is a subset of their Cartesian product A × B.
- If A = B, then the relation R on A is a subset of A × A.
- The total number of relations from a set consisting of m elements to a set consisting of n elements is 2^{mn}.
- If (a, b) belongs to R, then a is related to b and is written as ‘a R b’. If (a, b) does not belong to R, then a is not related to b and it is written as ‘a $\not R$ b’.

2. Co-domain and Range of a Relation

Let R be a relation from A to B. Then the ‘domain of $R' \subset A$ and the ‘range of $R' \subset B$. Co-domain is either set B or any of its superset or subset containing range of R.

3. Types of Relations

A relation R in a set A is called an empty relation if no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$.

A relation R in a set A is called a universal relation if each element of A is related to every element of A, i.e., R = A × A.

4. A relation R on a set A is called:

a. Reflexive, if (a, a) $\in$ R for every a $\in$ A.

b. Symmetric, if $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$ for all $a_1, a_2 \in A$.

c. Transitive, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that (a1, a3) $\in$ R for all $a_1, a_2, a_3 \in A$.

5. Equivalence Relation

- A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

- An empty relation R on a non-empty set X (i.e., 'a R b' is never true) is not an equivalence relation, because although it is vacuously symmetric and transitive, but it is not reflexive (except when X is also empty).

6. Given an arbitrary equivalence relation R in a set X, R divides X into mutually disjoint subsets S_i called partitions or subdivisions of X provided:

a. All elements of S, are related to each other for all i.

b. No element of Si is related to any element of St if i ≠ j.

c. $\bigcup_{i=1}^{n} S_j = X$ and $S_i \cap S_j = \phi$ if $i \neq j$.

The subsets St are called equivalence classes.

7. Union, Intersection and Inverse of Equivalence Relations

a. If R and S are two equivalence relations on a set A, R ∩ S is also an equivalence relation on A.

b. The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.

c. The inverse of an equivalence relation is an equivalence relation.

Top Concepts in Functions

1. Introduction to functions

A function from a non-empty set A to another non-empty set B is a correspondence or a rule which associates every element of A to a unique element of B written as $f : A \rightarrow B$ such that $f(x) = y$ for all $x \in A$, $y \in B$.

All functions are relations, but the converse is not true.

2. Domain, Co-domain and Range of a Function

- If $f : A \rightarrow B$ is a function, then set A is the domain, set B is the co-domain and set {f(x) : x ∈ A) is the range of f.
- The range is a subset of the co-domain.
- A function can also be regarded as a machine which gives a unique output in set B corresponding to each input from set A.

- If A and B are two sets having m and n elements, respectively, then the total number of functions from A to B is n^m.

3. Real Function

- A function $f : A \rightarrow B$ is called a real-valued function if B is a subset of R.
- If A and B both are subsets of R, then 'f' is called a real function.
- While describing real functions using mathematical formula, x (the input) is the independent variable and y (the output) is the dependent variable.
- The graph of a real function 'f' consists of points whose co-ordinates (x, y) satisfy $y = f(x)$, for all $x \in$ Domain(f).

4. Vertical line test

A curve in a plane represents the graph of a real function if and only if no vertical line intersects it more than once.

5. One-one Function

- A function $f : A \rightarrow B$ is one-to-one if for all $x, y \in A$, $f(x) = f(y) \Rightarrow x = y$ or $x \neq y \Rightarrow f(x) \neq f(y)$.
- A one-one function is known as an injection or injective function. Otherwise, f is called many-one.

6. Onto Function

- A function $f : A \rightarrow B$ is an onto function, if for each $b \in B$, there is at least one $a \in A$ such that $f(a) = b$, i.e., if every element in B is the image of some element in A, then f is an onto or surjective function.
- For an onto function, range = co-domain.
- A function which is both one-one and onto is called a bijective function or a bijection.
- A one-one function defined from a finite set to itself is always onto, but if the set is infinite, then it is not the case.

7. Let A and B be two finite sets and $f : A \rightarrow B$ be a function.

- If f is an injection, then $n(A) \leq n(B)$.
- If f is a surjection, then $n(A) \geq n(B)$.
- If f is a bijection, then $n(A) = n(B)$.

8. If A and B are two non-empty finite sets containing m and n elements, respectively, then

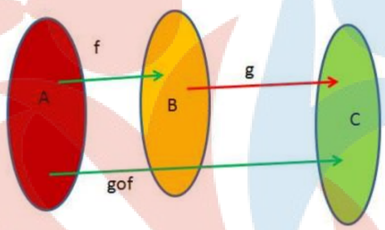
Number of functions from A to B = n^m.

- Number of one-one function from A to B $= \begin{cases} {}^nC_m \times m!, & \text{if } n \ge m \\ 0, & \text{if } n<m \end{cases}$
- Number of onto functions from A to B $= \begin{cases} \sum_{r=1}^{n} (-1)^{n-r}\ {}^nC_r r^m, & \text{if } m \ge n \\ 0, & \text{if } m<n \end{cases}$
- Number of one-one and onto functions from A to B $= \begin{cases} n!, & \text{if } m=n \\ 0, & \text{if } m \ne n \end{cases}$

9. If a function f : A → B is not an onto function, then f : A → f(A) is always an onto function.

10.Composition of Functions

Let f : A → B and g : B → C be two functions. The composition of f and g, denoted by g o f, is defined as the function g o f: A → C and is given by g o f(x): A → C defined by g o f(x) = g(f(x)) ∀ x ∈ A.



- Composition of f and g is written as g o f and not f o g.
- g o f is defined if the range of f ⊆ domain of g, and f o g is defined if the range of g ⊆ domain of f.
- Composition of functions is not commutative in general i.e., f o g(x) ≠ g o f(x).
- Composition is associative i.e., if f : X → Y, g : Y → Z and h : Z → S are functions, then h o (g o f) = (h o g) o f.
- The composition of two bijections is a bijection.

11.Inverse of a Function

- Let f : A → B is a bijection, then g : B → A is inverse of f if f(x) = y ⟺ g(y) = x OR g o f = I_A and f o g = I_B
- If g o f = I_A and f is an injection, then g is a surjection.
- If f o g' I_B and f is a surjection, then g is an injection.

12. Let f : A → B and g: B → C be two functions. Then

- g o f: A → C is onto ⇒ g: B → C is onto.
- g o f: A → C is one-one ⇒ f:A → B is one-one.

- $g \circ f: A \to C$ is onto and $g: B \to C$ is one-one $\Rightarrow f: A \to B$ is onto.
- $g \circ f: A \to C$ is one-one and $f: A \to B$ is onto $\Rightarrow g: B \to C$ is one-one.

13. Invertible Function

- A function $f : X \to Y$ is defined to be invertible if there exists a function $g : Y \to X$ such that gof - I_x and fog = I_y.
- The function g is called the inverse of f and is denoted by f^{-1}. If f is invertible, then f must be one-one and onto, and conversely, if f is one-one and onto, then f must be invertible.
- If $f : A \to B$ and $g : B \to C$ are one-one and onto, then $g \circ f : A \to C$ is also one-one and onto. But if g o f is one-one, then only f is one-one and g may or may not be one-one. If g o f is onto, then g is onto and f may or may not be onto.
- Let $f : X \to Y$ and $g : Y \to Z$ be two invertible functions. Then g o f is also invertible with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
- If $f: R \to R$ is invertible, $f(x) = y$, then $f^{-1}(y) = x$ and $(f^{-1})^{-1}$ is the function f itself.

Binary Operations

1. A binary operation * on a set A is a function from A × A to A.
2. If * is a binary operation on a set S, then S is closed with respect to *.
3. **Binary operations on R**
   - Addition, subtraction and multiplication are binary operations on R, which is the set of real numbers.
   - Division is not binary on R; however, division is a binary operation on R - {0} which is the set of non-zero real numbers.
4. **Laws of Binary Operations**
   - A binary operation * on the set X is called commutative, if $a * b = b * a$, for every $a, b \in X$.
   - A binary operation * on the set X is called associative, if $a (b*c) = (a*b)*c$, for every $a, b, c \in X$.
   - An element $e \in A$ is called an identity of A with respect to * if for each $a \in A$, $a * e = a = e * a$.
   - The identity element of (A, *) if it exists, is unique.
5. **Existence of Inverse**

   Given a binary operation * from $A \times A \to A$ with the identity element e in A, an element a e A is said to be invertible with respect to the operation *, if there exists an element b in

A such that a * b = e = b * a and b is called the inverse of a and is denoted by a^{-1}.

6. If the operation table is symmetric about the diagonal line, then the operation is commutative.

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

The operation * is commutative.

7. Binary Operation on Natural Numbers

Addition '+' and multiplication '-' on N, the set of natural numbers, are binary operations. However, subtraction '—' and division are not, because (4, 5) = 4 - 5 = -1 ∈ N and 4/5 = .8 ∈ N.

8. Number of Binary Operations

- Let S be a finite set consisting of n elements. Then S x S has n^2 elements.
- The total number of functions from a finite set A to a finite set B is $[n(B)]^{n(A)}$. Therefore, total number of binary operations on S is n^{n^2}.
- The total number of commutative binary operations on a set consisting of n elements is n $\frac{n(n-1)}{2}$.

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Chapter- 1 : Relations and Functions

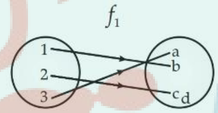
Relations and Functions

Types of functions

One - one (injective)

$f : x \to y$ is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ $\forall\ x_1, x_2 \in x$. Other wise, f is many- one, f_1 is one-one.

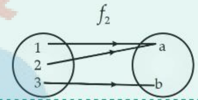
f_1



Onto (Surjective)

$f : x \to y$ is onto it for energy $y \in Y, \exists\ x \in X$ S.t. $f(x) = y$, f_2 is onto.

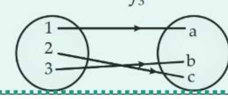
f_2



Bijective

$f : x \to y$ is both one-one and onto, then f is bijective. f_3 is bijective.

f_3



Composition of functions and Invertible functions

Invertible functions

A function $f : x \to y$ is invertible, if $\exists$ a function $g : y \to x$ such that $gof = I_X$ and $fog = I_Y$. Then, g is the inverse of f. If f is invertible, then it is both one-one and onto and vice-versa. For eg. If $f(x) = x$ and $f : N \to N$, then f is invertible.

Theorem 1 : If $f : x \to y$, $g : y \to z$ and $h : z \to s$ are functions, then $ho(got) = (hog)of$.

Theorem 2 : Let $f : x \to y$ and $g : y \to z$ be two invertible functions, then gof is invertible and $(gof)^{-1} = f^{-1} \text{ o } g^{-1}$.

Composition of functions

The composition of functions $f : A \to B$ and $g : B \to C$ is denoted by gof, and is defined as $gof : A \to C$ given by $gof(x) = g(f(x))\ \forall x \in A$. e.g. let $A = N$ and $f, g : N \to N$ such that $f(x) = x^2$ and $g(x) = x^3$ $\forall\ x \in N$. Then $gof(2) = g(f(2)) = g(2^2) = 4^3 = 64$.

Binary Operations Definition and its types

A binary operation '*' on a set A is a function $* : A \times A \to A$ denoted by $a * b$ i.e. $\forall$ $a, b, \in A$, $a * b \in A$. Commutative if $a * b = b * a\ \forall\ a, b \in A$. Associative if $(a * b) * c = a * (b * c)\ \forall a, b, \in A$. $e \in A$ is identity if $a * e = a = e * a\ \forall a \in A$. and $b \in A$ is the inverse of $a \in A$, if. $a * b = e = b * a$. Addition is a binary operation on the set of integers.

Types of Relations

Empty Relations Universal relation

A relation $R : A \to A$ is empty if $a \not R b\ \forall\ a, b \in A$. $R = \phi \subset A \times A$.
For eg : $R = \{(a,b) : a = b^2\}$, $A = \{1,5,10\}$

A relation $R : A \to A$ is universal if $a\ R\ b\ \forall, a,b \in A, R = A \times A$.
if $R = \phi$, then R is universal.

} Trivial Relations

Reflexive relation

A relation $R : A \to A$ is reflexive if $aRa\ \forall\ a \in A$

Symmetric Relation

A relation $R : A \to A$ is symmetric if $aRb \Rightarrow bRa\ \forall\ a, b \in A$

Transitive relation

A relation $R : A \times A$ is transitive if $aRb, bRc \Rightarrow aRc\ \forall\ a,b,c \in A$.

Equivalence relation

(reflexive, symmetric, transitive e.g., Let T = the set of all triangles in a plane and $R : T \to T$ defined by $R = \{(T_1, T_2)\} : T_1$ is congruent to $T_2\}$. Then, R is equivalence.

Important Questions

Multiple Choice questions-

1. Let R be the relation in the set (1, 2, 3, 4}, given by:

R = {(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)}.

Then:

(a) R is reflexive and symmetric but not transitive

(b) R is reflexive and transitive but not symmetric

(c) R is symmetric and transitive but not reflexive

(d) R is an equivalence relation.

2. Let R be the relation in the set N given by: R = {(a, b): a = b – 2, b > 6}. Then:

(a) (2, 4) ∈ R

(b) (3, 8) ∈ R

(c) (6, 8) ∈ R

(d) (8, 7) ∈ R.

3. Let A = {1, 2, 3}. Then number of relations containing {1, 2} and {1, 3}, which are reflexive and symmetric but not transitive is:

(a) 1

(b) 2

(c) 3

(d) 4.

4. Let A = (1, 2, 3). Then the number of equivalence relations containing (1, 2) is

(a) 1

(b) 2

(c) 3

(d) 4.

5. Let f: R → R be defined as $f(x) = x^4$. Then

(a) f is one-one onto

(b) f is many-one onto

(c) f is one-one but not onto

(d) f is neither one-one nor onto.

6. Let f: R → R be defined as f(x) = 3x. Then

(a) f is one-one onto

(b) f is many-one onto

(c) f is one-one but not onto

(d) f is neither one-one nor onto.

7. If f: R → R be given by $f(x) = (3 - x^3)^{1/3}$, then $f_o f$ (x) is

(a) $x^{1/3}$

(b) x^3

(c) x

(d) $3 - x^3$.

8. Let f: R – {$-\frac{4}{3}$} → R be a function defined as: $f(x) = \frac{4x}{3x+4}$, $x \neq -\frac{4}{3}$. The inverse of f is map g: Range f → R -{$-\frac{4}{3}$} given by

(a) $g(y) = \frac{3y}{3-4y}$

(b) $g(y) = \frac{4y}{4-3y}$

(c) $g(y) = \frac{4y}{3-4y}$

(d) $g(y) = \frac{3y}{4-3y}$

9. Let R be a relation on the set N of natural numbers defined by nRm if n divides m. Then R is

(a) Reflexive and symmetric

(b) Transitive and symmetric

(c) Equivalence

(d) Reflexive, transitive but not symmetric.

10. Set A has 3 elements, and the set B has 4 elements. Then the number of injective mappings that can be defined from A to B is:

(a) 144

(b) 12

(c) 24

(d) 64

Very Short Questions:

1. If R = {(x, y) : x + 2y = 8} is a relation in N, write the range of R.
2. Show that a one-one function:

   f {1, 2, 3} → {1, 2, 3} must be onto. (N.C.E.R.T.)
3. What is the range of the function $f(x) = \frac{|x-1|}{x-1}$? (C.B.S.E. 2010)
4. Show that the function f : N → N given by f(x) = 2x is one-one but not onto. (N.C.E.R.T.)
5. If f : R → R is defined by f(x) = 3x + 2 find f(f(x)). C.B.S.E. 2011 (F))
6. If $f(x) = \frac{x}{x-1}$, $x \neq 1$ then find fof. (N.C.E.R.T)
7. If f: R → R is defined by $f(x) = (3 - x^3)^{1/3}$, find fof (x)
8. Are f and q both necessarily onto, if gof is onto? (N.C.E.R.T.)

Short Questions:

1. Let A be the set of all students of a Boys' school. Show that the relation R in A given by:

   R = {(a, b): a is sister of b} is an empty relation and the relation R' given by :

   R' = {(a, b) : the difference between heights of a and b is less than 3 metres} is an universal relation. (N.C.E.R.T.)

2. Let $f : X \rightarrow Y$ be a function. Define a relation R in X given by :

   $R = \{(a,b):f(a) = f(b)\}$.

   Examine, if R is an equivalence relation. (N.C.E.R.T.)

3. Let R be the relation in the set Z of integers given by:

   $R = \{(a, b): 2 \text{ divides } a - b\}$.

   Show that the relation R is transitive. Write the equivalence class [0]. (C.B.S.E. Sample Paper 2019-20)

4. Show that the function:

   $f : N \rightarrow N$

   given by $f(1) = f(2) = 1$ and $f(x) = x - 1$, for every $x > 2$ is onto but not one-one. (N.C.E.R.T.)

5. Find gof and fog, if:

   $f : R \rightarrow R$ and $g : R \rightarrow R$ are given by $f(x) = \cos x$ and $g(x) = 3x^2$. Show that gof ≠ fog. (N. C.E.R. T.)

6. If $f(x) = \frac{4x+3}{6x-4}, x \neq \frac{2}{3}$ find fof(x)

7. Let $A = N \times N$ be the set of ail ordered pairs of natural numbers and R be the relation on the set A defined by (a, b) R (c, d) iff ad = bc. Show that R is an equivalence relation.

8. Let $f: R \rightarrow R$ be the Signum function defined as:

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

   and $g : R \rightarrow R$ be the Greatest Integer Function given by $g(x) = [x]$, where [x] is greatest integer less than or equal to x. Then does fog and gof coincide in (0,1]?

Long Questions:

1. Show that the relation R on R defined as $R = \{(a, b):a \leq b\}$, is reflexive and transitive but not symmetric.

2. Prove that function f : N → N, defined by $f(x) = x^2 + x + 1$ is one-one but not onto. Find inverse of f : N → S, where S is range of f.

3. Let A = {x ∈Z : 0 ≤ x ≤ 12}.

   Show that R = {(a, b) : a, b ∈ A; |a – b| is divisible by 4} is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class [2]. (C.B.S.E 2018)

4. Prove that the function f: [0, ∞) → R given by $f(x) = 9x^2 + 6x - 5$ is not invertible. Modify the co-domain of the function f to make it invertible, and hence find f-1. (C.B.S.E. Sample Paper 2018-19

Assertion and Reason Questions-

1. Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.
b) Both A and R are true but R is not the correct explanation of A.
c) A is true but R is false.
d) A is false and R is also false.

Assertion(A): Let L be the set of all lines in a plane and R be the relation in L defined as R = {(L1, L2): L1 is perpendicular to L2}.R is not equivalence realtion.

Reason (R): R is symmetric but neither reflexive nor transitive

2. Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.
b) Both A and R are true but R is not the correct explanation of A.
c) A is true but R is false.
d) A is false and R is also false.

Assertion (A): = {(T1, T2): T1 is congruent to T2}. Then R is an equivalence relation.

Reason(R): Any relation R is an equivalence relation, if it is reflexive, symmetric and transitive.

Case Study Questions-

1. Consider the mapping f: A → B is defined by f(x) = x - 1 such that f is a bijection.

Based on the above information, answer the following questions.

(i) Domain of f is:

a) R - {2}
b) R
c) R - {1, 2}
d) R - {0}

(ii) Range of f is:

a) R
b) R - {2}
c) R - {0}
d) R - {1, 2}

(iii) If g: R - {2} → R - {1} is defined by g(x) = 2f(x) - 1, then g(x) in terms of x is:

a. $\frac{x+2}{x}$

b. $\frac{x+1}{x-2}$

c. $\frac{x-2}{x}$

d. $\frac{x}{x-2}$

(iv) The function g defined above, is:

a) One-one
b) Many-one
c) into
d) None of these

(v)A function f(x) is said to be one-one if.

a. $f(x_1) = f(x_2) \Rightarrow -x_1 = x_2$
b. $f(-x_1) = f(-x_2) \Rightarrow -x_1 = x_2$
c. $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
d. None of these

2. A relation R on a set A is said to be an equivalence relation on A iff it is:

I. Reflexive i.e., $(a, a) \in R \ \forall \ a \in A$.
II. Symmetric i.e., $(a, b) \in R \Rightarrow (b, a) \in R \ \forall \ a, b \in A$.
III. Transitive i.e., $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R \ \forall \ a, b, c \in A$.

Based on the above information, answer the following questions.

(i) If the relation R = {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)} defined on the set A = {1, 2, 3}, then R is:

a) Reflexive
b) Symmetric
c) Transitive
d) Equivalence

(ii) If the relation R = {(1, 2), (2, 1), (1, 3), (3, 1)} defined on the set A = {1, 2, 3}, then R is:

a) Reflexive
b) Symmetric
c) Transitive
d) Equivalence

(iii) If the relation R on the set N of all natural numbers defined as R = {(x, y): y = x + 5 and x < 4}, then R is:

a) Reflexive
b) Symmetric
c) Transitive
d) Equivalence

(iv) If the relation R on the set A = {1, 2, 3, ………, 13, 14} defined as R = {(x, y): 3x - y = O}, then R is:

a) Reflexive
b) Symmetric
c) Transitive
d) Equivalence

(v) If the relation R on the set A = {I, 2, 3} defined as R = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}, then R is:

a) Reflexive only

b) Symmetric only
c) Transitive only
d) Equivalence

Answer Key-

Multiple Choice questions-

(b) R is reflexive and transitive but not symmetric

(c) $(6, 8) \in R$

(a) 1

(b) 2

(d) f is neither one-one nor onto.

(a) f is one-one onto

(c) x

(b) $g(y) = \frac{4y}{4-3y}$

(b) Transitive and symmetric

(c) 24

Very Short Answer:

1. Solution: Range of R = {1, 2, 3}.

   [$\because$ When x = 2, then y = 3, when x = 4, then y = 2, when x = 6, then y = 1]

2. Solution: Since 'f' is one-one,

   $\therefore$ under 'f', all the three elements of {1, 2, 3} should correspond to three different elements of the co-domain {1, 2, 3}.

   Hence, 'f' is onto.

3. Solution: When x > 1,

   than $f(x) = \frac{x-1}{x-1} = 1$.

   When x < 1,

than $f(x) = \frac{-(x-1)}{x-1} = -1$

Hence, Rf = {-1, 1}.

4. Solution:

Let $x_1, x_2 \in N$.

Now, $f(x_1) = f(x_2)$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

⇒ f is one-one.

Now, f is not onto.

∵ For $1 \in N$, there does not exist any $x \in N$ such that $f(x) = 2x = 1$.

Hence, f is ono-one but not onto.

5. Solution:

f(f(x)) = 3 f(x) + 2

= 3(3x + 2) + 2 = 9x + 8.

6. Solution:

$$fof(x) = f(f(x)) = \frac{f(x)}{f(x)-1}$$

$$= \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1} = \frac{x}{x-x+1}$$

$$= \frac{x}{1} = x.$$

7. Solution:

$f_o f(x) = f(f(x)) = (3-(f(x))^3)^{1/3}$

$= (3 - ((3 - x^3)^{1/3})^3)^{1/3}$

$= (3 - (3 - x^3))^{1/3} = (x^3)^{1/3} = x.$

8. Solution:

Consider f: {1, 2, 3, 4} → {1, 2, 3, 4}

and g: {1, 2, 3,4} → {1,2.3} defined by:

f(1) = 1, f(2) = 2, f(3) = f(4) = 3

g (1) = 1, g (2) = 2, g (3) = g (4) = 3.

∴ gof = g (f(x)) {1, 2,3}, which is onto

But f is not onto.

[∵ 4 is not the image of any element]

Short Answer:

1. Solution:

(i) Here R = {(a, b): a is sister of b}.

Since the school is a Boys' school,

∴ no student of the school can be the sister of any student of the school.

Thus R = Φ Hence, R is an empty relation.

(ii) Here R' = {(a,b): the difference between heights of a and b is less than 3 metres}.

Since the difference between heights of any two students of the school is to be less than 3 metres,

∴ R' = A x A. Hence, R' is a universal relation.

2. Solution:

For each a ∈ X, (a, a) ∈ R.

Thus R is reflexive. [∵ f (a) = f(a)]

Now (a, b) ∈ R

⇒ f(a) = f(b)

⇒ f(b) = f (a)

⇒ (b, a) ∈ R.

Thus R is symmetric.

And $(a, b) \in R$

and $(b, c) \in R$

$\Rightarrow f(a) = f(b)$

and $f(b) = f(c)$

$\Rightarrow f(a)= f(c)$

$\Rightarrow (a, c) \in R$.

Thus R is transitive.

Hence, R is an equivalence relation.

3. Solution:

Let 2 divide $(a - b)$ and 2 divide $(b - c)$, where $a,b,c \in Z$

$\Rightarrow$ 2 divides $[(a - b) + (b - c)]$

$\Rightarrow$ 2 divides $(a - c)$.

Hence, R is transitive.

And $[0] = \{0, \pm 2, \pm 4, \pm 6,...]$.

4. Solution:

Since $f(1) = f(2) = 1$,

$\therefore f(1) = f(2)$, where $1 \neq 2$.

$\therefore$ 'f' is not one-one.

Let $y \in N, y \neq 1$,

we can choose x as $y + 1$ such that $f(x) = x - 1$

$= y + 1 - 1 = y$.

Also $1 \in N, f(1) = 1$.

Thus 'f ' is onto.

Hence, 'f ' is onto but not one-one.

5. Solution:

We have:

f(x) = cos x and g(x) = $3x^2$.

∴ gof (x) = g (f(x)) = g (cos x)

= 3 $(\cos x)^2$ = 3 $\cos^2 x$

and fog (x) = f(g (x)) = f($3x^2$) = cos $3x^2$.

Hence, gof ≠ fog.

6. Solution:

We have: $\frac{4x+3}{6x-4}$ ...(1)

∴ fof(x) - f (f (x))

$$= \frac{4f(x)+3}{6f(x)-4}$$

$$= \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4} \quad [Using\ (1)]$$

$$= \frac{16x+12+18x-12}{24x+18-24x+16}$$

$$= \frac{34x}{34} = x.$$

7. Solution:

Given: (a, b) R (c, d) if and only if ad = bc.

(I) (a, b) R (a, b) iff ab – ba, which is true.

[∵ ab = ba ∀ a, b ∈ N]

Thus, R is reflexive.

(II) (a, b) R (c,d) ⇒ ad = bc

(c, d) R (a, b) ⇒ cb = da.

But cb = be and da = ad in N.

∴ (a, b) R (c, d) ⇒ (c, d) R (a, b).

Thus, R is symmetric.

(III) (a,b) R (c, d)

⇒ ad = bc ...(1)

(c, d) R (e,f)

⇒ cf = de ... (2)

Multiplying (1) and (2), (ad). (cf) – (be), (de)

⇒ af = be

⇒ (a,b) = R(e,f).

Thus, R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

8. Solution:

For x ∈ (0,1].

$$(fog)\,(x) = f(g(x)) = f([x])$$

$$= \begin{cases} f(0)\,;\ \text{if } 0 < x < 1 \\ f(1)\,;\ \text{if } x = 1 \end{cases}$$

$$\Rightarrow \quad f(g(x)) = \begin{cases} 0\,;\ \text{if } 0 < x < 1 \\ 1\,;\ \text{if } x = 1 \end{cases} \quad ...(1)$$

And (gof) (x) = g(f(x)) = g(1)

[∵ f(x) = 1 ∀ x > 0]

= [1] = 1

⇒ (gof) (x) = 1 ∀ x ∈ (0, 1] ...(2)

From (1) and (2), (fog) and (gof) do not coincide in (0, 1].

Long Answer:

1. Solution:

We have: R = {(a, b)} = a ≤ b}.

Since, a ≤ a ∀ a ∈ R,

∴ (a, a) ∈ R,

Thus, R reflexive.

Now, (a, b) ∈ R and (b, c) ∈ R

⇒ a ≤ b and b ≤ c

⇒ a ≤ c

⇒ (a, c) ∈ R.

Thus, R is transitive.

But R is not symmetric

[∵ (3, 5) ∈ R but (5, 3) ∉ R as 3 ≤ 5 but 5 > 3]

Solution:

Let $x_1, x_2 \in N$.

Now, $f(x_1) = f(x_2)$

$$\Rightarrow x^2{}_1 + x_1 + 1 = x^2{}_2 + x_2 + 1$$
$$\Rightarrow x^2{}_1 + x_1 = x^2{}_2 + x_2$$
$$\Rightarrow (x^2{}_1 - x^2{}_2) + (x_1 - x_2) = 0$$
$$\Rightarrow (x_1 - x_2) + (x_1 + x_2 + 1) = 0$$
$$\Rightarrow x_1 - x_2 = 0 \quad [\because x_1 + x_2 + 1 \neq 0]$$
$$\Rightarrow x_1 = x_2.$$

Thus, f is one-one.

Let y ∈ N, then for any x,

f(x) = y if y = $x^2 + x + 1$

$$\Rightarrow \quad y = \left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4}$$

$$\Rightarrow \quad y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow \quad x + \frac{1}{2} = \pm\sqrt{y - \frac{3}{4}}$$

$$\Rightarrow \quad x = \pm\frac{\sqrt{4y-3}}{2} - \frac{1}{2}$$

$$\Rightarrow \quad x = \frac{\pm\sqrt{4y-3} - 1}{2}$$

$$\Rightarrow \quad x = \frac{\sqrt{4y-3} - 1}{2}$$

$$\left[\frac{-\sqrt{4y-3} - 1}{2} \notin N \text{ for any value of } y\right]$$

Now, for y = $\frac{3}{4}$, x = $-\frac{1}{2} \notin$ N.

Thus, f is not onto.

⇒ f(x) is not invertible.

Since, x > 0, therefore, $\frac{\sqrt{4y-3}-1}{2} > 0$

$\Rightarrow \sqrt{4y-3} > 1$

⇒ 4y − 3 > 1

⇒ 4y > 4

⇒ y > 1.

Redefining, f : (0, ∞) ⟶ (1, ∞) makes

f(x) = x^2 + x + 1 on onto function.

Thus, f (x) is bijection, hence f is invertible and f^{-1} : (1, ∞) ⟶ (1,0)

$f^{-1}(y) = \frac{\sqrt{4y-3}-1}{2}$

2. Solution:

We have:

R = {(a, b): a, b ∈ A; |a – b| is divisible by 4}.

(1) Reflexive: For any a ∈ A,

∴ (a, b) ∈ R.

|a – a| = 0, which is divisible by 4.

Thus, R is reflexive.

Symmetric:

Let (a, b) ∈ R

⇒ |a – b| is divisible by 4

⇒ |b – a| is divisible by 4

Thus, R is symmetric.

Transitive: Let (a, b) ∈ R and (b, c) ∈ R

⇒ |a – b| is divisible by 4 and |b – c| is divisible by 4

⇒ $|a - b| = 4\lambda$

⇒ $a - b = \pm 4\lambda$ ....(1)

and $|b - c| = 4\mu$

⇒ $b - c = \pm 4\mu$ ....(2)

Adding (1) and (2),

$(a-b) + (b-c) = \pm 4(\lambda + \mu)$

⇒ $a - c = \pm 4(\lambda + \mu)$

⇒ (a, c) ∈ R.

Thus, R is transitive.

Now, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

(ii) Let 'x' be an element of A such that (x, 1) ∈ R

⇒ |x – 1| is divisible by 4

$\Rightarrow x - 1 = 0, 4, 8, 12, \ldots$

$\Rightarrow x = 1, 5, 9, 13, \ldots$

Hence, the set of all elements of A which are related to 1 is {1, 5, 9}.

(iii) Let $(x, 2) \in R$.

Thus $|x - 2| = 4k$, where $k \leq 3$.

$\therefore x = 2, 6, 10$.

Hence, equivalence class [2] = {2, 6, 10}.

3. Solution:

Let $y \in R$.

For any x, $f(x) = y$ if $y = 9x^2 + 6x - 5$

$\Rightarrow y = (9x^2 + 6x + 1) - 6$

$= (3x + 1)^2 - 6$

$$\Rightarrow \quad 3x + 1 = \pm\sqrt{y+6}$$

$$\Rightarrow \quad x = \frac{\pm\sqrt{y+6}-1}{3}$$

$$\Rightarrow \quad x = \frac{\sqrt{y+6}-1}{3}$$

$$\left[\because \frac{-\sqrt{y+6}-1}{3} \notin [0, \infty) \text{ for any value of } y\right]$$

For $y = -6 \in R$, $x = -\frac{1}{3} \notin [0, \infty)$.

Thus, f(x) is not onto.

Hence, f(x) is not invertible.

Since, $x \geq 0$, $\therefore \frac{\sqrt{y+6}-1}{3} \geq 0$

$\Rightarrow \sqrt{y+6}-1 \geq 0$

$\Rightarrow \sqrt{y+6} \geq 1$

$\Rightarrow y+6 \geq 1$

$\Rightarrow y \geq -5.$

We redefine,

f: $[0, \infty) \rightarrow [-5, \infty)$,

which makes $f(x) = 9x^2 + 6x - 5$ an onto function.

Now, $x_1, x_2 \in [0, \infty)$ such that $f(x_1) = f(x_2)$

$\Rightarrow (3x_1 + 1)^2 = (3x_2 + 1)^2$

$\Rightarrow [(3x_1 + 1) + (3x_2 + 1)][(3x_1 + 1) - (3x_2 + 1)]$

$\Rightarrow [3(x_1 + x_2) + 2][3(x_1 - x_2)] = 0$

$\Rightarrow x_1 = x_2$

$[\because 3(x_1 + x_2) + 2 > 0]$

Thus, f(x) is one-one.

$\therefore$ f(x) is bijective, hence f is invertible

and $f^{-1}: [-5, \infty) \rightarrow [0, \infty)$

$f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$

Assertion and Reason Answers-

1. (a) Both A and R are true and R is the correct explanation of A.

2. (a) Both A and R are true and R is the correct explanation of A.

Case Study Answers-

1. Answer :

(i) (a) R - {2}

Solution:

For f(x) to be defined x - 2; ≠ 0 i.e., x; ≠ 2.

$\therefore$ Domain of f = R - {2}

(ii) (b) R - {2}

Solution:

Let y = f(x), then $y = \frac{x-1}{x-2}$

⇒ xy - 2y = x - 1 ⇒ xy - x = 2y -

$$\Rightarrow x = \frac{2y-1}{y-1}$$

Since, x ∈∈ R - {2}, therefore y ≠ 1

Hence, range of f = R - {1}

(iii) (d) $\frac{x}{x-2}$

Solution:

We have, g(x) = 2f(x) - 1

$$= 2\left(\frac{x-1}{x-2}\right) - 1 = \frac{2x-2-x+2}{x-2} = \frac{x}{x-2}$$

(iv) (a) One-one

Solution:

We have, $g(x) = \frac{x}{x-2}$

Let $g(x_1) = g(x_2) \Rightarrow \frac{x_1}{x_1-2} = \frac{x_2}{x_2-2}$

⇒ $x_1x_2 - 2x_1 = x_1x_2 - 2x_2 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

Thus, $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$

Hence, g(x) is one-one.

(v)(c) $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

2. Answer :

(i) (a) Reflexive

Solution:

Clearly, (1, 1), (2, 2), (3, 3), ∈ R. So, R is reflexive on A.

Since, (1, 2) ∈ R but (2, 1) ∉ R. So, R is not symmetric on A.

Since, (2, 3), ∈ R and (3, 1) ∈ R but (2, 1) ∉ R. So, R is not transitive on A.

(ii) (b) Symmetric

Solution:

Since, (1, 1), (2, 2) and (3, 3) are not in R. So, R is not reflexive on A.

Now, (1, 2) ∈ R ⇒ (2, 1) ∈ R and (1, 3) ∈ R ⇒ (3, 1) ∈ R. So, R is symmetric,

Clearly, (1, 2) ∈ R and (2, 1) ∈ R but (1, 1) ∉ R. So, R is not transitive on A.

(iii) (c) Transitive

Solution:

We have, R = {(x, y): y = x + 5 and x < 4}, where x, y ∈ N.

∴ R = {(1, 6), (2, 7), (3, 8)}

Clearly, (1, 1), (2, 2) etc. are not in R. So, R is not reflexive.

Since, (1, 6) ∈ R but (6, 1) ∉ R. So, R is not symmetric.

Since, (1, 6) ∈ R and there is no order pair in R which has 6 as the first element.

Same is the case for (2, 7) and (3, 8). So, R is transitive.

(iv) (d) Equivalence

Solution:

We have, R = {(x, y): 3x - y = 0}, where x, y ∈ A = {1,2, ..., 14}.

∴ R = {(1, 3), (2, 6), (3, 9), (4, 12)}

Clearly, (1, 1) ∉ R. So, R is not reflexive on A.

Since, (1, 3) ∈ R but (3, 1) ∉ R. So, R is not symmetric on A.

Since, $(1, 3) \in R$ and $(3, 9) \in R$ but $(1, 9) \notin R$. So, R is not transitive on A.

(v)(d) Equi0076alence

Solution:

Clearly, $(1, 1), (2, 2), (3, 3) \in R$. So, R is reflexive on A.

We find that the ordered pairs obtained by interchanging the components of ordered pairs in R are also in R. So, R is symmetric on A. For $1, 2, 3 \in A$ such that (1, 2) and (2, 3) are in R implies that (1, 3) is also, in R. So, R is transitive on A. Thus, R is an equivalence relation.